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A Study on Fuzzy Number

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Abstract— In this article, an alternative method to evaluate the arithmetic operations on fuzzy number has been developed, on the assumption that the **Dubois- Prade** left and right reference functions of a fuzzy number are ***distribution function*** and ***complementary distribution function*** respectively. This alternative method has been demonstrated with the help of numerical examples. Lastly; computations are done using Matlab.

Index Terms— distribution functions ,complementary distribution functions, membership function, fuzzy set ,valuation set ,Arithmetic of Fuzzy Number, Fuzzy Number.

1. Introduction

Fuzziness means different things depending upon the domain of application and the way, it is measured. By means of fuzzy sets, usual notions can be described mathematically in their very abstractness. Fuzzy set theory has been widely acclaimed as offering greater richness in applications than ordinary set theory.

Among the various types of fuzzy sets, those which are defined on the universal set R of real numbers are of particular importance. They may, under certain conditions, be viewed as fuzzy numbers, which reflect the human perception of uncertain numerical quantification. After the successful applications of fuzzy sets theory on the controller systems, this theory have applied in other areas. In the most these applications fuzzy numbers are one way to describe the data vagueness and imprecision. They can be regarded as an extension of the real numbers.

Fuzzy numbers are of great importance in fuzzy systems. In the applications, the triangular and the trapezoidal fuzzy numbers are usually used. It is now a vigorous area of research with manifold applications.

2. Preliminaries

2.1.0 Definition[9]: A function $\mu_A : X \to [0,1]$ is called a ***membership function***.

2.1.1 Definition[9]: A function $f : X \longrightarrow [0,1]$ is called a ***fuzzy set*** on X, where X is a nonempty set of objects called ***referential set*** and [0,1] (the unit interval) is called ***valuation set*** and $\forall x \in X$; $f(x)$ represents the grade of membership of x. we shall write the symbol I for [0,1].

2.1.1(a) Example: Fuzzy sets with a discrete non-ordered universe:

Let X = {San Francisco, Boston, Los Angeles} be the set of cities one may choose to live in.The fuzzy set A = "desirable city to live in" may be described as follows:

A = {(San Francisco, 0.9), (Boston, 0.8), (Los Angeles, 0.6)}.

Apparently the universe of discourse X is discrete and it contains non ordered objects - in this case, three big cities in the United States. As one can see, the foregoing membership grades listed above are quite subjective; anyone can come up with three different but legitimate values to reflect his or her preference.

2.1.1(b) Example: Fuzzy sets with a discrete ordered universe:

Let X = {0, 1, 2, 3, 4, 5, 6} be the set of numbers of children a family may choose to have.

Then the fuzzy set B = "desirable number of children in a family" may be described as follows:

B = {(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)}. Here we have a discrete ordered universe X.

2.1.1(c) Example: Suppose V is collection of vowels in English alphabets. Here the collection is well defined, as which alphabet is vowel and which is not vowel, we can specify. Then V is a classical set.

3. Fuzzy Number

3.1 Fuzzy Number [9]: To qualify a fuzzy number, a fuzzy set A on R must possess at least the following three properties:

1. A must be a normal fuzzy set.
2. A^α must be a closed interval for every $\alpha \in (0,1]$.
3. The support of A, A^{0+} must be bounded.

4. Arithmetic of Fuzzy Number

4.1.1 Addition of Fuzzy Numbers [2]: Consider $X = [a,b,c]$ and $Y = [p,q,r]$ be two triangular fuzzy numbers. Suppose $Z = X + Y = [a+p, b+q, c+r]$ be the fuzzy number of $X + Y$.

Let fmf of X and Y be $\mu_X(x)$ and $\mu_Y(y)$ as mentioned below:

$$\mu_X(x) = \begin{cases} L(x), a \le x \le b \\ R(x), b \le x \le c \\ 0, otherwise \end{cases} \quad (1)$$

and

$$\mu_Y(y) = \begin{cases} L(y), a \le y \le b \\ R(y), b \le y \le c \\ 0, otherwise \end{cases} \quad (2)$$

Where $L(x)$ and $L(y)$ are the left reference functions and $R(x)$ and $R(y)$ are the right reference functions respectively. We assume that

$L(x)$ and $L(y)$ are distribution function and $R(x)$ and $R(y)$ are complementary distribution function. Accordingly, there would exist some density functions for the distribution functions $L(x)$ and (1-$R(x)$).

Let $$f(x)=\frac{d}{dx}(L(x)), a\le x\le b \text{ and}$$
$$g(x)=\frac{d}{dx}(1-R(x)), b\le x\le c.$$

We start with equating $L(x)$ with $L(y)$, and $R(x)$ with $R(y)$. And so, we obtain

$$y=\phi_1(x) \text{ and}$$
$$y=\phi_2(x) \text{ respectively.}$$

Let $z=x+y$, so we have $z=x+\phi_1(x)$ and $z=x+\phi_2(x)$, so that $x=\psi_1(z)$ and $x=\psi_2(z)$ (say). Replacing *x* by $\psi_1(z)$ in *f*(*x*), and by $\psi_2(z)$ in *g*(*x*), we obtain $f(x)=\eta_1(z)$ and $g(x)=\eta_2(z)$ (say).

Now let $$\frac{dx}{dz}=\frac{d}{dz}(\psi_1(z))=m_1(z)$$

and $$\frac{dx}{dz}=\frac{d}{dz}(\psi_2(z))=m_2(z)$$

The distribution function for *X*+*Y*, would now be given by

$$\int_{a+p}^{x}\eta_1(z)m_1(z)dz,\ \ a+p\le x\le b+q.$$

and the complimentary distribution function would be given by

$$1-\int_{b+q}^{x}\eta_2(z)m_2(z)dz,\ \ b+q\le x\le c+r.$$

We claim that this distribution function and the complementary distribution function constitute the fuzzy membership function of *X*+*Y* as follows:

$$\mu_{X+Y}(x)=\begin{cases}\int_{a+p}^{x}\eta_1(z)m_1(z)dz, & a+p\le x\le b+q\\ 1-\int_{b+q}^{x}\eta_2(z)m_2(z)dz, & b+q\le x\le c+r\\ 0 & ,otherwise.\end{cases}$$

4.1.1Example: Let $X=[1,2,4]$ and $Y=[3,5,6]$ be two triangular fuzzy numbers with fmf

$$\mu_X(x)=\begin{cases} x-1, 1\le x\le 2\\ \frac{4-x}{2},\ 2\le x\le 4\\ 0, otherwise\end{cases}$$

and $$\mu_Y(y)=\begin{cases}\frac{y-3}{2},\ 3\le y\le 5\\ 6-y,\ 5\le y\le 6\\ 0, otherwise\end{cases}$$

Here *X*+*Y*=[4,7,10]. Equating the distribution function and complementary distribution function, we obtain

$$y=\phi_1(x)=2x+1$$

and $$y=\phi_2(x)=\frac{x+8}{2}.$$

Let $z=x+y$, so we have

$$z=x+\phi_1(x)=3x+1$$

and $$z=x+\phi_2(x)=\frac{3x+8}{2}$$

so that $$x=\psi_1(z)=\frac{z-1}{3}$$

and $$x=\psi_2(z)=\frac{2z-8}{3} \text{ respectively.}$$

Replacing *x* by $\psi_1(z)$ in *f*(*x*), and by $\psi_2(z)$ in *g*(*x*), we obtain $f(x)=1=\eta_1(z)$ and $g(x)=\frac{1}{2}=\eta_2(z)$.

Now let $$m_1(z)=\frac{d}{dz}(\psi_1(z))=\frac{1}{3}$$

and $$m_2(z)=\frac{d}{dz}(\psi_2(z))=\frac{2}{3}$$

Then the fmf of *X*+*Y* would be given by

$$\mu_{X+Y}(x)=\begin{cases}\frac{x-4}{3},\ 4\le x\le 7\\ \frac{10-x}{3},\ 7\le x\le 10\\ 0 \quad ,otherwise.\end{cases}$$

Matlab Function:

```
point_n = 100;
min_x = -10; max_x = 30;
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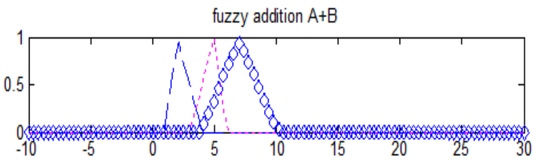
```
x = linspace(min_x, max_x, point_n)';
A = trimf(x, [1 2 4]);
B = trimf(x, [3 5 6]);
C = fuzarith(x, A, B, 'sum');
subplot(4,1,1);
plot(x, A, 'b--', x, B, 'm:', x, C, 'd');
title('fuzzy addition A+B');
```

Graph:



4.1.2 Subtraction of Fuzzy Numbers [2]: Suppose $X=[a,b,c]$ and $Y=[p,q,r]$ be two triangular fuzzy numbers. Let fmf of X and Y be $\mu_X(x)$ and $\mu_Y(y)$ as mentioned below:

$$\mu_X(x)=\begin{cases} L(x), a \le x \le b \\ R(x), b \le x \le c \\ 0, otherwise \end{cases} \qquad (1)$$

and $$\mu_Y(y)=\begin{cases} L(y), a \le y \le b \\ R(y), b \le y \le c \\ 0, otherwise \end{cases} \qquad (2)$$

Suppose *Z=X-Y* .Then the fuzzy membership function of *Z=X-Y* would be given by
Z=X+(-Y).

Suppose $(-Y)=[-r,-q,-p]$ be the fuzzy number of $(-Y)$. We assume that $L(y)$ and $R(y)$ as distribution function and complementary distribution function respectively. Accordingly, there would exist some density functions for the distribution functions $L(y)$ and (1- $R(y)$).

Let $$f(y)=\frac{d}{dy}(L(y)), p \le y \le q$$

and $$g(y)=\frac{d}{dy}(1-R(y)), q \le y \le r .$$

Now let $t=-y$ so that

$$\frac{dy}{dt}=-1=m(t), \text{ (say).}$$

Replacing $y=-t$ in $f(y)$ and $g(y)$, we obtain

$$f(y)=\eta_1(t)$$

and $$g(y)=\eta_2(t) \text{ ,(say).}$$

Then the fmf of $(-Y)$ would be given by

$$\mu_{-Y}(y)=\begin{cases} \int_{-r}^{y}\eta_2(t)m(t)dt, & -r \le y \le -q \\ 1-\int_{-q}^{y}\eta_1(t)m(t)dt, & -q \le y \le -p \\ 0 & , otherwise. \end{cases}$$

Then we can easily find the fmf of $X-Y$ by addition of fuzzy numbers X and $(-Y)$ as described in the earlier section.

4.1.2 Example: Let $X=[1,2,4]$ and $Y=[3,5,6]$ be two triangular fuzzy numbers with fmf

$$\mu_X(x)=\begin{cases} x-1, 1 \le x \le 2 \\ \frac{4-x}{2}, \ 2 \le x \le 4 \\ 0, otherwise \end{cases}$$

and $$\mu_Y(y)=\begin{cases} \frac{y-3}{2}, \ 3 \le y \le 5 \\ 6-y, \ 5 \le y \le 6 \\ 0, otherwise \end{cases}$$

Suppose, $Z=X-Y$ or $Z=X+(-Y)$.

Now, $-Y=[-6,-5,-3]$ be the fuzzy number of $(-Y)$.

Let $t=-y$ so that $y=-t$, which implies $m(t)=-1$. Then the density function $f(y)$ and $g(y)$ would be say,

$$f(y)=\frac{d}{dy}(\frac{y-3}{2})=\frac{1}{2}=\eta_1(t), \ 3 \le y \le 5$$

and

$$g(y)=\frac{d}{dy}(1-(6-y))=1=\eta_2(t), \ 5 \le y \le 6$$

Then the fmf of $(-Y)$ would be given by

$$\mu_{-Y}(y)=\begin{cases} \frac{6+y}{6-5}, & -6 \le y \le -5 \\ \frac{3+y}{3-5}, & -5 \le y \le -3 \\ 0 & , otherwise. \end{cases}$$

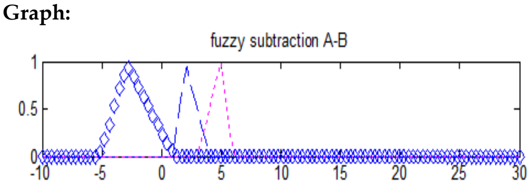
Then by addition of fuzzy numbers $X=[1,2,4]$ and $-Y=[-6,-5,-3]$ the fmf of $X-Y$ is given by,

$$\mu_{X+(-Y)}(x)=\begin{cases}\dfrac{x+5}{2}, & -5\le x\le -3\\ \dfrac{1-x}{4}, & -3\le x\le 1\\ 0 & ,otherwise.\end{cases}$$

Matlab Function:

```
point_n = 100;
min_x = -10; max_x = 30;
x = linspace(min_x, max_x, point_n)';
A = trimf(x, [1 2 4]);
B = trimf(x, [3 5 6]);
C = fuzarith(x, A, B, 'sub');
subplot(4,1,1);
plot(x, A, 'b--', x, B, 'm:', x, C, 'd');
title('fuzzy subtraction A-B');
```

Graph:



4.1.3 Multiplication of Fuzzy Numbers [2]: Let $X=[a,b,c],\ (a,b,c>0)$ and $Y=[p,q,r],\ (p,q,r>0)$ be two triangular fuzzy numbers with fuzzy membership functions. Let fmf of X and Y be $\mu_X(x)$ and $\mu_Y(y)$ as mentioned below:

$$\mu_X(x)=\begin{cases}L(x), a\le x\le b\\ R(x), b\le x\le c\\ 0, otherwise\end{cases} \quad (1)$$

and

$$\mu_Y(y)=\begin{cases}L(y), a\le y\le b\\ R(y), b\le y\le c\\ 0, otherwise\end{cases} \quad (2)$$

Suppose $Z=X.Y=[a.p,b.q,c.r]$ be the fuzzy number of $X.Y$. $L(x)$ and $L(y)$ are the left reference functions and $R(x)$ and $R(y)$ are the right reference functions respectively. We assume that $L(x)$ and $L(y)$ are distribution function and $R(x)$ and $R(y)$ are complementary distribution function. Accordingly, there would exist some density functions for the distribution functions $L(x)$ and (1-$R(x)$).

Let $$f(x)=\frac{d}{dx}(L(x)), a\le x\le b$$

and $$g(x)=\frac{d}{dx}(1-R(x)), b\le x\le c.$$

We again start with equating $L(x)$ with $L(y)$, and $R(x)$ with $R(y)$. And so, we obtain

$$y=\phi_1(x)$$

and $$y=\phi_2(x)$$ respectively.

Let $z=x.y$, so we have

$$z=x.\phi_1(x)$$

and $$z=x.\phi_2(x),$$

so that $$x=\psi_1(z)$$

and $$x=\psi_2(z)$$ (say).

Replacing x by $\psi_1(z)$ in $f(x)$, and by $\psi_2(z)$ in $g(x)$, we obtain

$$f(x)=\eta_1(z) \text{ and } g(x)=\eta_2(z) \text{ say.}$$

Now let, $$\frac{dx}{dz}=\frac{d}{dz}(\psi_1(z))=m_1(z)$$

and $$\frac{dx}{dz}=\frac{d}{dz}(\psi_2(z))=m_2(z)$$

The distribution function for $X.Y$, would now be given by

$$\int_{ap}^{x}\eta_1(z)m_1(z)dz,\ ap\le x\le bq.$$

And the complimentary distribution function would be given by

$$1-\int_{bq}^{x}\eta_2(z)m_2(z)dz,\ bq\le x\le cr.$$

We are claiming that this distribution function and the complementary distribution function constitute the fuzzy membership function of $X.Y$ as follows:

$$\mu_{X.Y}(x)=\begin{cases}\displaystyle\int_{ap}^{x}\eta_1(z)m_1(z)dz, & ap\le x\le bq\\ 1-\displaystyle\int_{bq}^{x}\eta_2(z)m_2(z)dz, & bq\le x\le cr\\ 0 & ,otherwise.\end{cases}$$

4.1.3 Example: Let $X=[1,2,4]$ and $Y=[3,5,6]$ be two triangular fuzzy numbers with fmf

$$\mu_X(x)=\begin{cases}x-1, 1\le x\le 2\\ \dfrac{4-x}{2}, & 2\le x\le 4\\ 0, otherwise\end{cases}$$

and

$$\mu_Y(y)=\begin{cases}\dfrac{y-3}{2}, & 3\le y\le 5\\ 6-y, & 5\le y\le 6\\ 0, otherwise\end{cases}$$

Here $X.Y$=[3,10,24]. Equating the distribution function and complementary distribution function, we obtain

$$y=\phi_1(x)=2x+1$$

and $$y=\phi_2(x)=\frac{x+8}{2}.$$

Let $z=x.y$, so we shall have $z=x.\phi_1(x)=2x^2+x$

and $$z=x.\phi_2(x)=\frac{8x+x^2}{2},$$

so that

$$x=\psi_1(z)=\frac{-1\pm\sqrt{1+8z}}{4}$$

and $$x=\psi_2(z)=-4\pm\sqrt{16+2z}.$$

Replacing x by $\psi_1(z)$ in $f(x)$ and by $\psi_2(z)$ in $g(x)$, we obtain

$$f(x)=1=\eta_1(z)$$

and $$g(x)=\frac{1}{2}=\eta_2(z).$$

Now let, $$m_1(z)=\frac{d}{dz}(\psi_1(z))$$

and $$m_2(z)=\frac{d}{dz}(\psi_2(z)).$$

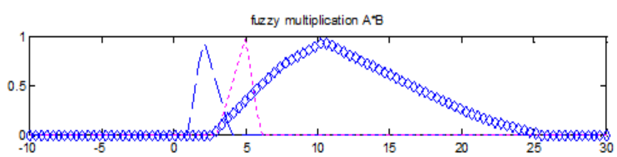
Then the fmf of $X.Y$ would be given by,

$$\mu_{X\cdot Y}(x)=\begin{cases}\dfrac{\sqrt{1+8x}-5}{4},\ 3\le x\le 10\\ \dfrac{8-\sqrt{16+2x}}{2},\ 10\le x\le 24\\ 0 \quad ,otherwise.\end{cases}$$

Matlab Function:

```
point_n = 100;
min_x = -10; max_x = 30;
x = linspace(min_x, max_x, point_n)';
A = trimf(x, [1 2 4]);
B = trimf(x, [3 5 6]);
C= fuzarith(x, A, B, 'prod');
subplot(4,1,1);
plot(x, A, 'b--', x, B, 'm:', x, C, 'd');
title('fuzzy multiplication A*B');
```

Graph:



4.1.4 Division of Fuzzy Numbers [2]: Let $X=[a,b,c],\ (a,b,c>0)$ and $Y=[p,q,r],\ (p,q,r>0)$ be two triangular fuzzy numbers with fuzzy membership functions. Let fmf of X and Y be $\mu_X(x)$ and $\mu_Y(y)$ as mentioned below:

$$\mu_X(x)=\begin{cases}L(x), a\le x\le b\\ R(x), b\le x\le c\\ 0, otherwise\end{cases} \qquad (1)$$

and $$\mu_Y(y)=\begin{cases}L(y), a\le y\le b\\ R(y), b\le y\le c\\ 0, otherwise\end{cases} \qquad (2)$$

Suppose $Z=\frac{X}{Y}$.Then the fuzzy membership function of $Z=\frac{X}{Y}$ would be given by $Z=X.Y^{-1}$.

At first, we have to find the fmf of Y^{-1}. Suppose $Y^{-1}=(r^{-1},q^{-1},p^{-1})$ be the fuzzy number of Y^{-1}. We assume that $L(y)$ and $R(y)$ as distribution function and complementary distribution function respectively. Accordingly, there would exist some density functions for the distribution functions $L(y)$ and (1- $R(y)$).

Say, $$f(y)=\frac{d}{dy}(L(y)),\ p\le y\le q$$ and $$g(y)=\frac{d}{dy}(1-R(y)), q\le y\le r.$$

Let $t=y^{-1}$

so that $$\frac{dy}{dt}=-\frac{1}{t^2}=m(t)\text{ ,(say).}$$

Replacing $y=t^{-1}$ in $f(y)$ and $g(y)$ we obtain

$$f(y)=\eta_1(t)$$

and $$g(y)=\eta_2(t)\text{ ,(say).}$$

Then the fmf of (Y^{-1}) would be given by

$$\mu_{Y^{-1}}(y)=\begin{cases}\displaystyle\int_{r^{-1}}^{y}\eta_2(t)m(t)dt,\ r^{-1}\le y\le q^{-1}\\ 1-\displaystyle\int_{q^{-1}}^{y}\eta_1(t)m(t)dt,\ q^{-1}\le y\le p^{-1}\\ 0 \quad ,otherwise\end{cases}$$

Next we can easily find the find the fmf of $\frac{X}{Y}$ by multiplication of fuzzy numbers X and Y^{-1} as described in the earlier section.

4.1.4 Example: Let $X=[1,2,4]$ and $Y=[3,5,6]$ be two triangular fuzzy numbers with fmf

$$\mu_X(x)=\begin{cases} x-1, 1\le x\le 2 \\ \dfrac{4-x}{2},\ 2\le x\le 4 \\ 0, otherwise \end{cases}$$

and
$$\mu_Y(y)=\begin{cases} \dfrac{y-3}{2},\ 3\le y\le 5 \\ 6-y,\ 5\le y\le 6 \\ 0, otherwise \end{cases}$$

Suppose $Z=\dfrac{X}{Y}$ or $Z=X.Y^{-1}$.

Then the fmf of $Y^{-1}=[6^{-1},5^{-1},3^{-1}]$ is given by

$$\mu_{\frac{1}{Y}}(y)=\begin{cases} \dfrac{6-\frac{1}{y}}{6-5},\ \dfrac{1}{6}\le y\le\dfrac{1}{5} \\ \dfrac{\frac{1}{y}-3}{5-3},\ \dfrac{1}{5}\le y\le\dfrac{1}{3} \\ 0\quad , otherwise \end{cases}$$

Then by multiplication of fuzzy numbers $X=[1,2,4]$

and $Y^{-1}=[6^{-1},5^{-1},3^{-1}]$

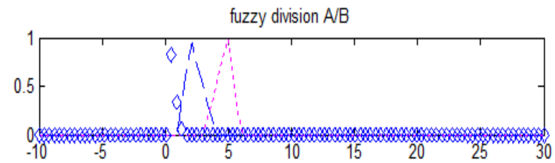
The membership function of $\dfrac{X}{Y}$ would be given by

$$\mu_{\frac{X}{Y}}(x)=\begin{cases} \dfrac{6x-1}{x+1},\ \dfrac{1}{6}\le x\le\dfrac{2}{5} \\ \dfrac{4-3x}{2(x+1)},\ \dfrac{2}{5}\le x\le\dfrac{4}{3} \\ 0\quad , otherwise. \end{cases}$$

Matlab Function:

```
point_n = 100;
min_x = -10; max_x = 30;
x = linspace(min_x, max_x, point_n)';
A = trimf(x, [1 2 4]);
B = trimf(x, [3 5 6]);
C= fuzarith(x, A, B, 'div');
subplot(4,1,1);
plot(x, A, 'b--', x, B, 'm:', x, C4, 'd');
title('fuzzy division A/B');
```

Graph:



5. Conclusion

The standard method of α -cuts to the membership of a fuzzy number does not always yield results . We have demonstrated that an assumption that the **Dubois-Prade** left reference function is a distribution function and that the right reference function is a complementary distribution function leads to a very simple way of dealing with fuzzy arithmetic. Further, this alternative method can be utilized in the cases where the method of α -cuts fails, e.g. in finding the fmf of $\sqrt{X}$.

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